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EFFECTIVE PERMEABILITY OF A HIGHLY POROUS MEDIUM

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The problem of the effective conductivity of a medium with a low concentration of inclusions has been treated in many papers (e.g., [1]). The case of a medium with a random distribution of circular inclusions characterized by a binary correlation function was treated in [2] by using the apparatus of ensemble averages. We use methods of the theory of functions of a complex variable to solve the two-dimensional problem of the effective permeability of a medium with translational symmetry of an arrangement of circular inclusions. Since a correlation function does not have to be defined for an ordered arrangement of inclusions, the effective permeability of the medium can be determined when the concentration of inclusions is not low. By using methods of the theory of functions of a complex variable, we obtain an effective solution of this kind of problem for inclusions of arbitrary shape by conformal mapping onto the exterior of a unit circle. In this sense the solution of the basic problem is reduced. The problem was solved by using the approach developed in [3, 4] for determining the state of stress of a plane weakened by an infinite number of circular holes. The basic idea of this approach consists in representing the required solution in the form of a Laurent series by expanding it in terms of the small parameter $\varepsilon = 1/l$, where l is the distance between centers of the inclusions, and using the basic idea of the Bubnov–Galerkin method to find the expansion coefficients. As in the elasticity problem, this is an effective method of solving transmissibility problems in a medium with an infinite number of inclusions. By averaging the solution over a macroscopic volume the effective transmissibility coefficient of such a medium can be determined.

Filtration in a Medium with Circular Inclusions. Let us consider the steady filtration of a fluid in a medium with circular inclusions arranged as shown in Fig. 1. Without loss of generality, we take the inclusions of unit radius. The distances along the x and y axes between the centers of neighboring circles are assumed equal to l. Thus, the centers of the circles lie at the points

$$z_{n,p} = l(n + ip),$$

where $i = \sqrt{-1}$; $n = 0, \pm 1, \pm 2, \ldots, \pm\infty$; $p = 0, \pm 2, \ldots \pm\infty$. As in [5], it is convenient to describe filtration flow by introducing the complex potentials

$$\varphi_\nu = (k_\nu/\mu)P_\nu + i\psi_\nu,\ \nu = 0,\ 1.$$

Here φ_0 corresponds to the filtration region in the medium outside an inclusion, and φ_1 to the region inside an inclusion; the k_ν are the permeabilities of the medium and inclusion, respectively; μ is the viscosity of the fluid; the P_ν are the pressures of the fluid in the medium and within an inclusion respectively; the ψ_ν are the flow functions. The complex potentials must satisfy Laplace's equation

$$\Delta\varphi_\nu = 0 \tag{1}$$

and are analytic in the respective domains of definition. In addition, the joining conditions

$$\frac{\partial}{\partial n_1}\operatorname{Re}\varphi_0 = \frac{\partial}{\partial n_1}\operatorname{Re}\varphi_1; \tag{2}$$

$$\frac{\partial}{\partial s}\operatorname{Re}\frac{1}{k_0}\varphi_0 = \frac{\partial}{\partial s}\operatorname{Re}\frac{1}{k_1}\varphi_1. \tag{3}$$

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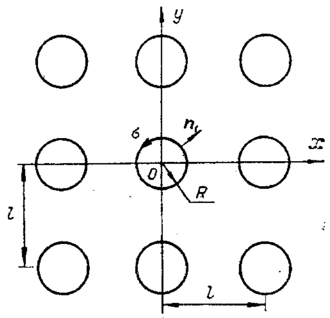

Fig. 1

must be satisfied on the boundary of an inclusion. Here the first equation represents the conservation of flux along the normal n_1 to the boundary of the inclusion, and the second equation is obtained from the equality of pressures on the contour of an inclusion after differentiating this condition with respect to the coordinate s measured along the contour. The expression Re $\varphi_\nu = {}^1\!/_2(\varphi_\nu + \overline{\varphi}_\nu)$ denotes the real part of the potential. From now on a bar over a quantity denotes its complex conjugate. Performing the differerentiation and adding Eqs. (2) and (3) gives a relation on the contour of a circular inclusion of unit radius [5]:

$$\sigma\varphi_0'(\sigma) = \frac{1+\alpha}{2}\sigma\varphi_1'(\sigma) + \frac{1-\alpha}{1}\overline{\sigma\varphi_1'(\sigma)}, \qquad (4)$$

where $\sigma = \exp(i\theta)$ and $\alpha = k_0/k_1$.

Similar conditions must be satisfied on the contour of any inclusion. In addition, the real parts of the potentials must be bounded in their domains of definition and satisfy a condition at infinity. When the flux density at infinity u_0 is specified parallel to the x axis, the potential φ_0 can be written in the form

$$\varphi_0 = u_0 z + \varphi_0^1,$$

where the function φ_0^1 is analytic outside the inclusions. The potentials being sought can be written in the form of Laurent series, which, taking account of the above, have the form

$$\varphi_0 = u_0 z + \sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{k=1}^{\infty}\frac{a_k}{[z - l(n+ip)]^k}; \qquad (5)$$

$$\varphi_1 = \sum_{k=1}^{\infty} b_k z^k. \qquad (6)$$

It is interesting to note that the expression for the potential φ_0 is in essence the sum of plane multipoles located at the centers of the inclusions.

Thus, the solution of the problem is reduced to the determination of the unknown coefficients a_k and b_k, which can be found from Eq. (4). It follows from the symmetry of the problem that if the potentials φ_0 and φ_1 satisfy (4) on one contour, they will automatically satisfy this condition on any other contour $(\sigma - z_{n,p})$. Therefore, it is sufficient to consider the case $\sigma = \exp(i\theta)$. We note that it also follows from the symmetry of the problem that the even coefficients a_k and b_k must vanish.

Equation (5) for the potential contains the series expansion parameter $\varepsilon = 1/l < 1/2$. This considerably simplifies the procedure for determining the coefficients by using the basic idea of the Bubnov–Galerkin method. We expand in series in powers of ε terms of the type $[z - (1/\varepsilon)(n + ip)]^{-k}$ of potential (5), and substitute this expansion and the expression for the potential φ_1 into the condition on the contour (4). As a result we have

$$F(\sigma) = \sigma\varphi_0'(\sigma) - \frac{1+\alpha}{2}\sigma\varphi_1'(\sigma) - \frac{1-\alpha}{2}\overline{\sigma\varphi_1'(\sigma)} = 0.$$

By requiring that the function $F(\sigma)$ be orthogonal to the set of functions $\sigma^{\pm k}$ ($k = 0, 1, 2, \ldots$), we obtain an infinite system of algebraic equations for a_k and b_k. By retaining a finite number of terms in the sums (5) and (6) it is possible to obtain a finite system of algebraic equations and an approximate solution of the problem with an accuracy which increases with an increase in the order of the approximation. For example, in the second approximation $k = 1, 2, 3$, the system of equations for a_k and b_k has the form

$$[(1-\alpha)/2]b_1 = -a_1,\quad [(1-\alpha)/2]b_3 = -a_3,$$
$$[(1+\alpha)/2]b_1 = u_0 - 2a_1\lambda_2\varepsilon^2 - 12a_3\lambda_5\varepsilon^5,\quad [(1+\alpha)/2]b_3 = -a_1\lambda_4\varepsilon^4 - 10a_3\lambda_6\varepsilon^6,$$

where $\lambda_t = \sum^{*}_{n=-\infty,\,p=-\infty}^{n=\infty,\,p=\infty} \left(\frac{1}{n+ip}\right)^t$. Here the asterisk on the summation sign denotes the omission of the zero term $n = p = 0$ in the sum. It should be noted that the coefficients λ_t decrease rapidly with increasing t, and successive approximations converge well even for $\varepsilon \to 1/2$.

It is of interest to consider the limit as the inclusions in the medium become more widely spaced ($\varepsilon \ll 1/2$) and absolutely impenetrable ($\alpha \to \infty$). In this case $\varphi_1 \to b_1 z \to 0$, since $b_1 \to 0$ as $\alpha \to \infty$, and $\varphi_0 \to u_0 z + a_1/z$, where $a_1 = u_0(\alpha - 1)/(\alpha + 1)$. Thus, in the limit the solution agrees with the well-known solution of the problem of the flow of an incompressible fluid around a cylinder [5]: $\varphi_1 = 0$, $\varphi_0 = u_0(1 + 1/z)$.

For a uniform medium ($\alpha = 1$), as should be expected, the solution determines the potential flow of a fluid with velocity u_0. It should be noted that if the concentration of the inclusions is not low, u_0 loses its physical meaning. In order to show this, let us calculate $\langle q\rangle$, the magnitude of filtration flow averaged over a macroscopic volume. It follows from the symmetry of the problem that

$$\langle q\rangle = \frac{1}{l/2}\int_0^{l/2} \frac{\partial}{\partial x}\operatorname{Re}\varphi_0\Big|_{x=l/2} dy.$$

Using (5), we obtain in the third approximation

$$\langle q\rangle = u_0 - 2a_1\beta_1\varepsilon^2 - 2a_3\beta_2\varepsilon^4, \tag{7}$$

where

$$\beta_1 = \sum_{n,p} \frac{\left(\frac{1}{2} - p\right)}{\left(\frac{1}{2} - p\right)^2 + \left(\frac{1}{2} - n\right)^2} \approx 1.57;\quad a_1 = \frac{\alpha - 1}{2} b_1;$$

$$\beta_2 = \sum_{n,p} \frac{\left(\frac{1}{2} - p\right)\left[3\left(\frac{1}{2} - n\right)^2 - \left(\frac{1}{2} - p\right)^2\right]}{\left[\left(\frac{1}{2} - n\right)^2 + \left(\frac{1}{2} - p\right)^2\right]^3} \approx 1{,}65\cdot 10^{-2};\quad a_3 = \frac{\alpha - 1}{2} b_3.$$

It is clear from (7) that for low concentrations of inclusions ($\varepsilon \to 0$), u_0 agrees with $\langle q\rangle$ and represents the average flux density in the medium. For a highly porous medium ($\alpha = 0$) the values of the $a_k < 0$, and the average flux density is larger than u_0, which loses its original physical meaning. At the same time the potentials (5) and (6) satisfy the solution of problem (1)-(3) but correspond to the case when the average flux density is determined by (7).

Effective Permeability of a Medium. For practical purposes it is very important to know the effective permeability of a medium. Averaging over a macroscopic volume V can be performed most simply by using the relation given in [1]

$$\frac{1}{V}\int \left(q - \frac{k_0}{\mu}\nabla P\right) dv = \langle q\rangle - \frac{k_0}{\mu}\langle \nabla P\rangle,$$

where q(z) is the filtration flux outside and inside the inclusions: $\langle q\rangle$ and $\langle \nabla P\rangle$ are the values of the flux and pressure gradient averaged over a macroscopic volume. The integrand is different from zero only inside the inclusions, whose concentration is N. As a result, the average value of the pressure gradient is

$$\langle \nabla P\rangle = \frac{\mu}{k_0}\left[\langle q\rangle - (k_1 - k_0)\frac{LN}{k_1}\right].$$

Here $L = \int(\partial/\partial x)\operatorname{Re}\varphi_1 dv$, where the integration is not extended over the whole macroscopic volume, but only with one of the inclusions. Accordingly, the effective permeability of the medium is determined with the formula $\langle K\rangle = \mu\langle q\rangle / \langle \nabla P\rangle$, and has the form

$$\langle K\rangle = k_0\left[1 - \frac{(k_1 - k_0)}{k_1\langle q\rangle} LN\right]^{-1}, \tag{8}$$

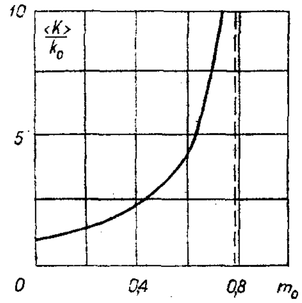

Fig. 2

where $\langle q\rangle$ is determined by (7). Thus, to calculate the effective permeability it is necessary to know the values of the coefficients in (6). In the first approximation, retaining only the terms with k = 1 in (5) and (6), we obtain $b_1 = 2\mu_0/(\alpha + 1)$, and the expression for the effective permeability is

$$\langle K\rangle = k_0\left[1 - 2\frac{k_1 - k_0}{k_1 + k_0}\frac{N\pi R^2}{\langle q\rangle}\right]^{-1}, \tag{9}$$

where R is the radius of an inclusion. For $N\pi R^2 \ll 1$, Eq. (9) gives the well-known formula for the effective transmissibility of a medium with widely spaced independent circular inclusions:

$$\langle K\rangle = k_0\left[1 + 2\frac{k_1 - k_0}{k_1 + k_0}N\pi R^2\right].$$

Successive approximations lead to a refinement of Eq. (9) and to an expression for the case when the concentration of the inclusions is not low. It is easy to show that for successive approximations $L = b_1\pi R^2$, but the value of b_1 is changed: $b_1 = 2u_0(\alpha+1)\left[(\alpha+1)^2 - 3(\alpha-1)^2\lambda_4^2\varepsilon^8\right]^{-1}$ in the second approximation, and $b_1 = 2u_0(\alpha+1) \times \left[(\alpha+1)^2 - \frac{3(\alpha-1)^2\lambda_4^2\varepsilon^8(\alpha+1)^2}{(\alpha+1)^2 - 735\lambda_8^2(\alpha-1)^2\varepsilon^{16}}\right]^{-1}$ in the third approximation. Here $\varepsilon = 1/l = R(N^{1/2})$; $\lambda_4 \approx 3.247$; $\lambda_8 \approx 2.068$.

It is of interest to consider the limiting case of a highly porous medium ($\alpha \to 0$). The formula corresponding to this case has the form

$$\langle K\rangle = k_0\left\{1 - \frac{2m_0}{\left[1 - 0.313m_0^4\left(1 - 0.331m_0^8\right)^{-1}\right]\left[1 + 0.999m_0 + 0.0021m_0^4\right]}\right\}^{-1}, \tag{10}$$

where $m_0 = \pi R^2 N$ is the porosity of the medium.

Figure 2 shows $\langle K\rangle/k_0$, the reduced effective permeability of the medium, as a function of its porosity. It is clear from the graph that the familiar linear relation (8) gives a good description of the effective permeability only for low porosity media ($m_0 < 0.1$). As $m_0 \to \pi/4$ the value of $\langle K\rangle/k_0 \to \infty$, and for $m_0 = \pi/4$ the inclusions touch one another and form an infinite cluster. In this case there is a jump in the transmissibility. In a system with a random distribution of inclusions, such a cluster is formed for appreciably smaller values of the porosity [6].

It should be noted that Eq. (10) was derived by assuming the linear Darcy law, which does not hold for high filtration velocities. In this case the filtration can be described by a two-term equation of the form [7]

$$\nabla P = (\mu/\langle K\rangle)v + (\rho/k_\rho)v^2, \tag{11}$$

where ρ is the density of the fluid and v is its velocity. When the second term on the right-hand side of Eq. (11) is small in comparison with the first, Eq. (11) goes over into the linear Darcy law. With an increase in the filtration velocity the term quadratic in the velocity increases more rapidly than the linear term, and from a certain value v_k it becomes dominant. In the equation describing the filtration of a fluid with higher velocities, it is necessary to take account of inertial terms.

It is of interest to estimate the limits of applicability of the linear Darcy law for describing filtration in a highly porous medium. To do this it is necessary to know k_ρ in Eq. (11). We estimate it by using the approach developed in [7]. According to [7] the quadratic term is related to ∇P_ρ, the pressure gradient necessary to overcome the filtration resistance related to the structure of the medium – the constriction and dilation of the pore channels. The linear term is related to the pressure gradient ∇P_μ necessary to overcome internal friction of the viscous fluid and friction on the walls of the pore channels. The total pressure gradient is determined by the sum of the gradients ∇P_ρ and ∇P_μ.

The pressure drop ∇P_ρ is related to the sudden broadening of a jet of fluid according to the Borda–Carnot theorem, given by the relation

$$\Delta P_\rho = (\rho/2)(u_n - u_b)^2,$$

where u_n is the average flow velocity in the narrow part of the channel, and u_b is the average flow velocity in the broadened part of the channel.

Taking account of the fact that the characteristic distance between pores in the medium is lR, the average value of the pressure gradient ∇P_ρ can be estimated in the form

$$\nabla P_\rho \approx (\rho/2lR)(u_n - u_b)^2. \tag{12}$$

For steady flow $(u_n - u_b)$

$$u_n - u_b = \langle q\rangle(1/S_n - 1/S_b), \tag{13}$$

where S_n and S_b are the areas of the openings of the pore channels in the narrow and broadened parts of the flow respectively. Introducing $\langle 1/S\rangle$ (the average value of $1/S(x)$, where $S(x)$ is the area of the opening of the pore channels in the cross section with coordinate x), the average value of the flux density in the medium can be written in the form

$$\langle q\rangle = (v/m_0)\langle 1/S\rangle. \tag{14}$$

Substituting (13) and (14) into (12) and comparing the result with Eq. (11), we obtain

$$k_\rho = 2lRm_0^2\left(\frac{S_n S_b}{S_n + S_b}\left\langle\frac{1}{S}\right\rangle\right)^2. \tag{15}$$

We introduce a quantity m_n characterizing the porosity of the medium outside the pores. Then

$$S_n = m_n lR.$$

The quantity S_b corresponds to the cross section passing through the center of the pore:

$$S_b = 2[1 + (l - 1)m_n]R.$$

By averaging $1/S(x)$ over an elementary volume containing the pore, we obtain

$$\left\langle\frac{1}{S}\right\rangle = \frac{1 - 2\varepsilon}{Rlm_n} + \frac{1}{Rl(1 - m_n)}\left[\frac{\pi}{2} - \frac{a}{\sqrt{1 - a^2}}\ln\frac{(a + 1 - \sqrt{1 - a^2})(1 + \sqrt{1 - a^2})}{(a + 1 + \sqrt{1 - a^2})(1 - \sqrt{1 - a^2})}\right],$$

where $a = lm_n/2(1 - m_n)$. Substituting the expressions obtained for S_n, S_b, and $\langle 1/S\rangle$ into (15), we obtain the final expression for k_ρ, which for $lm_n \ll 1$ has the form

$$k_\rho \approx 2Rlm_0^2\left\{1 - 2\varepsilon + m_n\frac{\pi}{2}\right\}^2. \tag{16}$$

Let us estimate the filtration velocity v_k for which the linear and quadratic terms in the two-term Eq. (11) are equal. Using (16) we find

$$v_k = \frac{2\mu Rlm_0^2}{\langle K\rangle\rho}\left(1 - 2\varepsilon + m_n\frac{\pi}{2}\right)^2. \tag{17}$$

Substituting into (17) $\mu = 10^{-2}$ P, $Rl = 10^{-3}$ m, $m_0 = 0.636$, $m_n = 10^{-2}$, $\varepsilon = 0.45$, $k_0 = 10^{-1}$ D, and determining $\langle K\rangle/k_0$ from Fig. 2, we obtain $v_k \approx 10$ m/sec.

Thus, the increase of the porosity of a medium as $m_0 \to \pi/4$ leads to a sharp increase in the effective permeability $\langle K\rangle$, which in turn for a fixed pressure drop leads to an increase in the filtration velocity and an increase in the quadratic term in the two-term filtration equation. As a result, starting from a certain value of the velocity, the quadratic term becomes dominant, and a further increase in $\langle K\rangle$ has practically no effect on the filtration velocity.

The relations we have derived enable us to determine not only the effective transmissibility of a medium with circular inclusions of arbitrary concentration, but also to obtain an analytic solution of the two-dimensional problem of filtration in a medium with translational symmetry of inclusions. The analytic solution can be obtained with any accuracy. The solution we have presented was obtained in the third approximation with an accuracy of $\sim \varepsilon^8$. The method used can also be applied to solve transmissibility problems in a medium with inclusions of arbitrary shape. It is very interesting that when the concentration of the inclusions is not low, the quantity u_0, characterizing the flux density at infinity for a problem with widely spaced inclusions, loses its original physical meaning, and the solution obtained corresponds to filtration of an average flux density different from u_0. We note that a similar effect occurs also in treating transport processes in a crystal lattice. This fact must also be taken into account in treating elasticity problems, for whose solution the method used in the present article was originally developed.

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THERMAL BOUNDARY LAYER ON A CYLINDRICAL GAS COLUMN WITH DISTRIBUTED HEAT SOURCES

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UDC 553.6.011

The study of stream interaction with a gas domain where energy liberation occurs is of practical and theoretical interest. We speak of problems when the leaking gas passes through the heat liberation space. The situation mentioned can occur in meterology, in stream heating in an electric arc or other form of electrical discharge, in the air cooling of stabilized gas heat-liberating elements in reactors, in powerful electron beam or other kinds of penetrating radiation propagation in a gas medium, etc. However, systematic computations of the flow and heat-transfer patterns have been executed in application to conditions for longitudinally air-cooled stabilized arcs. Their results are shown most completely in [1-4]. Semiempirical numerical [2, 4] and integral [1, 3] methods were used. There is also a number of theoretical papers of general nature on flows with distributed heat supply (see [5] and the citations there) and a cycle of investigations devoted to laser beam propagation and discharges on a substance (see [6]) which are primarily of estimating nature in the theoretical part.

The purpose of this paper is to compute the thermal boundary layer being formed during air cooling of a cylindrical gas column with arbitrary volume heat sources by an unbounded stream. The stationary problem is examined under the assumption that the main heat elimination mechanism is heat conduction. We limit ourselves to the case of longitudinal blowing around the column of heat liberating gas. In the reference system coupled to the free stream the problem is formulated differently: determine the perturbation of the gas state by the moving distributed heat sources.

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